An analytic equation of state for Ising-like models

Denjoe O'Connor[1]**, J A Santiago**[2] **and C R Stephens**[3]

[1] School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Republic of Ireland
[2] Centro de Investigación Avanzada en Ingeniería Industrial. Universidad Autónoma del Estado de Hidalgo, Pachuca 42184, Mexico
[3] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, México DF 04510, Mexico

E-mail: denjoe@stp.dias.ie, sgarciaj@uaeh.reduaeh.mx and stephens@nucleares.unam.mx

Abstract
Using an environmentally friendly renormalization we derive, from an underlying field theory representation, a formal expression for the equation of state, $y = f(x)$, that exhibits all desired asymptotic and analyticity properties in the three limits $x \to 0$, $x \to \infty$ and $x \to -1$. The only necessary inputs are the Wilson functions γ_λ, γ_φ and γ_{φ^2}, associated with a renormalization of the transverse vertex functions. These Wilson functions exhibit a crossover between the Wilson–Fisher fixed point and the fixed point that controls the coexistence curve. Restricting to the case $N = 1$, we derive a one-loop equation of state for $2 < d < 4$ naturally parameterized by a ratio of nonlinear scaling fields. For $d = 3$ we show that a non-parameterized analytic form can be deduced. Various asymptotic amplitudes are calculated directly from the equation of state in all three asymptotic limits of interest and comparison made with known results. By positing a scaling form for the equation of state inspired by the one-loop result, but adjusted to fit the known values of the critical exponents, we obtain better agreement with known asymptotic amplitudes.

PACS number: 64.60.Ak

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The universal equation of state for the Landau–Ginzburg–Wilson $O(N)$ model remains a subject of great interest (see, for instance, [1, 11] for recent reviews). Its calculation from first principles is much more difficult than the calculation of other universal quantities, such as critical exponents or amplitude ratios, as it exhibits crossover behaviour between three

distinct, asymptotic regimes—the critical region when approached from the critical isotherm, or when approached along the critical isochor, and the coexistence curve. Maintaining the correct analyticity properties of the equation of state in all these three distinct regimes has not been possible within the confines of first principle calculations, such as from a field-theoretic, microscopic Hamiltonian. Rather, appeal has been made to a parameterized phenomenological scaling ansatz [13] that exhibits the right asymptotics, the underlying microscopic theory then being used to fix various free parameters that exist in the ansatz.

In this paper we use a renormalization group methodology—environmentally friendly renormalization [9]—based only on an underlying Landau–Ginzburg–Wilson Hamiltonian and without the need for any phenomenological ansatz, to derive an equation of state that exhibits all desired analyticity properties in the three distinct asymptotic regimes.

2. The equation of state

As first noted by Widom [7], in the critical region, the equation of state is a homogeneous function relating an external magnetic field H, the reduced temperature t, and the magnetization φ, which can be expressed by

$$y = f(x) \tag{1}$$

where $f(x)$ is universal, the scaling variables y and x are $y = B_c^\delta H / \varphi^\delta$ and $x = B^{1/\beta} t / \varphi^{1/\beta}$, and B_c and B are non-universal amplitudes associated with the behaviour on the critical isotherm $t = 0$ and the coexistence curve $t < 0$, $H = 0$.

The scaling function $f(x)$ is normalized such that $f(0) = 1$ on the critical isotherm, and $f(-1) = 0$ on the coexistence curve. Several properties of the universal equation of state are known rigorously. For instance, it is known that $f(x)$ has a regular Taylor expansion around the limit $x = 0$ given by

$$f(x) = 1 + \sum_{n=1}^{\infty} f_n^0 x^n, \tag{2}$$

while in the limit $x \to \infty$, by Griffith's analyticity, one has an expansion of the form

$$f(x) = x^\gamma \sum_{n=0}^{\infty} f_n^\infty x^{-2n\beta}. \tag{3}$$

In the limit $x \to \infty$ a natural variable is $z = b_1 \varphi / t^\beta$, where $b_1 = \left(-C_4^+ / (C^+)^3\right)^{1/2}$ and the C_{2n}^+ are the amplitudes of the $2n$-point correlation functions for $T > T_c$, C^+ being the amplitude of the susceptibility. In terms of z the equation of state takes the form

$$H = -\left(-C^+ / C_4^+\right)^{1/2} t^{\beta\delta} F(z), \tag{4}$$

where the universal scaling function $F(z)$ for small z has an expansion of the form

$$F(z) = z + \frac{1}{6} z^3 + \sum_{n=3}^{\infty} \frac{r_{2n}}{(2n-1)!} z^{2n-1}, \tag{5}$$

where $r_2 = r_4 = 1$ by choice of normalization. As (5) is an expansion in φ, the constants r_{2n} are related to the $2n$-point correlation functions at $\varphi = 0$ and hence are very natural observables to calculate in lattice simulations. In the limit $z \to \infty$, $F(z)$ has an expansion of the form

$$F(z) = z^\delta \sum_{k=0}^{\infty} F_k^\infty z^{-k/\beta}. \tag{6}$$

The universal scaling functions $f(x)$ and $F(z)$ are related via

$$z^{-\delta} F(z) = F_0^\infty f(x) \tag{7}$$

with $z = z_0 x^{-\beta}$, where z_0 is the universal zero of the equation of state in terms of the variable z. Hence, the expansion coefficients of the two functions can be related to find

$$f_n^\infty = z_0^{2n+1-\delta} \frac{r_{2n+2}}{F_0^\infty (2n+1)!} \tag{8}$$

$$f_n^0 = \frac{F_n^\infty}{F_0^\infty} z_0^{-n/\beta}. \tag{9}$$

Thus, we see it is sufficient to know the expansion coefficients of $f(x)$ in the limits $x \to 0$ and ∞ in order to calculate the asymptotic properties of $F(z)$ and the interesting coefficients r_{2n}.

Unlike the limits $x \to 0$ and ∞, near the coexistence curve, $x \to -1$, there are no rigorous mathematical arguments as to the analyticity properties of $f(x)$, although there do exist conjectures. For instance, for $N > 1$ in [15], based on an ε-expansion analysis, it was conjectured that $(1 + x)$ has a double expansion in powers of y and $y^{(d-2)/2}$ of the form

$$1 + x = c_1 y + c_2 y^{1-\epsilon/2} + d_1 y^2 + d_2 y^{2-\epsilon/2} + \cdots. \tag{10}$$

In three dimensions it predicts an expansion of $(1 + x)$ in powers of $y^{1/2}$.

Of course, the behaviour in the vicinity of the coexistence curve depends on the value of N. For $N = 1$, the longitudinal correlation length remains finite away from the critical point on the coexistence curve, while for $N > 1$, the existence of Goldstone bosons leads to infrared singularities. For $N = 1$, one may formally posit that in the vicinity of the coexistence curve

$$f(x) = \sum_{n=1}^{\infty} f_n^c (1 + x)^n \tag{11}$$

the integer powers being a reflection of the finite longitudinal correlation length. For Ising-like systems essential singularities are to be expected. Of course, these cannot be captured within the confines of an ansatz like (11). For $N > 1$ studies of the nonlinear σ model lead one to expect a leading behaviour of the form

$$f(x) \sim c_f (1 + x)^{2/(d-2)} \tag{12}$$

though, as mentioned, the nature of the corrections to this behaviour is not well understood, although (10) is one conjecture. In the $1/N$ expansion there is some evidence [11] for logarithmic corrections of the form $\ln(1 + x)$ in three dimensions.

Early field theoretic calculations using the renormalization group with an ε-expansion [14] foundered on the fact that they did not exhibit Griffiths analyticity in the large x limit. Irrespective of the expansion method ε, fixed dimension $1/N$ etc—there will remain a fundamental difficulty—that an expansion around a particular fixed point will not readily access the universal equation of state in the entire phase diagram, due to the presence of other fixed points that must be accessed. Simply put, the equation of state exhibits crossovers, and the nature of these crossovers depends on N. For $N = 1$ the theory is controlled by a 'Gaussian' or mean-field fixed point[4], wherein fluctuations are suppressed on the coexistence curve away from the critical point by the non-vanishing longitudinal mass. In distinction, for $N > 1$, the theory is dominated by the massless Goldstone excitations on the coexistence curve and the nonlinear σ model gives a good description [2].

---

[4] Note however that this regime also corresponds to that of the 'strong-coupling discontinuity' fixed point [3, 4].

The problem of incorporating Griffiths analyticity was solved by using a parameterized formulation in terms of new variables R and θ, related to t and φ via

$$\varphi = m_0 R^\beta m(\theta) \qquad t = R(1 - \theta^2) \qquad H = h_0 R^{\beta\delta} h(\theta),$$

where the two functions $m(\theta)$ and $h(\theta)$ are undetermined. In this new parameterization the scaling variable x and the scaling function $f(x)$ are given by

$$x = \frac{1 - \theta^2}{\theta_0^2 - 1} \left(\frac{m(\theta_0)}{m(\theta)} \right)^{1/\beta} \tag{13}$$

$$f(x) = \left(\frac{m(\theta)}{m(1)} \right)^{-\delta} \frac{h(\theta)}{h(1)}. \tag{14}$$

Most current field theoretic formulations for determining the equation of state (see [1, 11] for comprehensive reviews) rely on such formulations. The drawback is that the underlying microscopic theory is not used to determine the functional form of $m(\theta)$ and $h(\theta)$, rather an ansatz is made as to the general functional form which depends on certain unknown parameters and then the underlying microscopic theory is used to fix these parameters. The most common ansatz is that the functions are polynomials in θ. The coefficients of the powers of θ are then determined by calculating certain observables independently from the underlying microscopic theory and then using the values of these observables to determine the coefficients. Various methodologies have been used [1, 11, 12] using a variety of methods. For instance, an expansion of the effective potential for small φ for $t > 0$ using ϵ or fixed-dimension expansions. Results from Monte Carlo simulations or high temperature expansions have also been used.

In this paper we apply a new method to describe the equation of state in a uniform manner. Using environmentally friendly renormalization [9], we find a schema able to capture the crossover between the Wilson–Fisher fixed point and the fixed point that controls the coexistence curve. By integrating along curves of constant magnetization, we obtain the equation of state in the whole critical region. Moreover, the equation of state that we obtain is parameterized in terms of the inverse of the transverse correlation length, a quantity well defined over the entire phase diagram. The representation we have found is valid for both large and small values of the scaling variables and satisfies Griffiths analyticity. Although our approach works for any N we here concentrate on the case $N = 1$.

3. A renormalization group representation of the equation of state

In this section we briefly outline the derivation of the equation of state for a theory described by the standard LGW Hamiltonian with $O(N)$ symmetry

$$\mathcal{H}[\varphi] = \int d^d x \left(\frac{1}{2} \nabla \varphi^a \nabla \varphi^a + \frac{1}{2} r(x) \varphi^a \varphi^a + \frac{\lambda_B}{4!} (\varphi^a \varphi^a)^2 \right) \tag{15}$$

with $r = r_c + t_B$, where r_c is the value of r at the critical temperature T_c and $t_B = \Lambda^2 \frac{(T - T_c)}{T_c}$, Λ being the microscopic scale. We denote a generic vertex function by $\Gamma_{l...lt...t}^{(N,M)}$, where the number of l and t subscripts indicates whether a longitudinal or a transverse propagator is to be attached to the vertex at the corresponding point. When all subscripts are either l or t we will use a single l or t, for example $\Gamma_{t...t}^{(N,M)}$ will be abbreviated $\Gamma_t^{(N,M)}$. Furthermore, when there are no φ^2 insertions (i.e. $M = 0$) the second index will be left off, e.g., $\Gamma^{(N)}$ indicates $\Gamma^{(N,0)}$.

Due to the Ward identities of the model, it is sufficient to know only the $\Gamma_t^{(N,M)}$, as all the other vertex functions can be reconstructed from these. The equation of state is

$$H = \Gamma_t^{(2)} \varphi. \tag{16}$$

For the Ising system equation (16) can be viewed as the definition of $\Gamma_t^{(2)}$. The remaining transverse vertex functions $\Gamma_t^{(N,M)}$ can similarly be identified. The resulting expressions are equivalent to an analytic continuation of the Ward identities (see [9] for details).

3.1. Renormalization in terms of nonlinear scaling fields

Due to the existence of large fluctuations in the critical regime a renormalization of the microscopic bare parameters of the form

$$t(m, \kappa) = Z_{\varphi^2}^{-1}(\kappa) t_B(m) \tag{17}$$

$$\lambda(\kappa) = Z_\lambda(\kappa) \lambda_B \tag{18}$$

$$\varphi(\kappa) = Z_\varphi^{-1/2}(\kappa) \varphi_B \tag{19}$$

must be imposed, where κ is an arbitrary renormalization scale. The renormalized parameters satisfy the differential equations

$$\kappa \frac{dt(\kappa)}{d\kappa} = \gamma_{\varphi^2}(\kappa) t(\kappa) \tag{20}$$

$$\kappa \frac{d\lambda(\kappa)}{d\kappa} = \gamma_\lambda(\kappa) \lambda(\kappa) \tag{21}$$

$$\kappa \frac{d\varphi(\kappa)}{d\kappa} = -\frac{1}{2} \gamma_\varphi(\kappa) \varphi(\kappa) \tag{22}$$

where the Wilson functions associated with this coordinate transformation are

$$\gamma_{\varphi^2}(\kappa) = -\kappa \frac{d}{d\kappa} \ln Z_{\varphi^2} \Big|_c \tag{23}$$

$$\gamma_\lambda(\kappa) = \kappa \frac{d}{d\kappa} \ln Z_\lambda \Big|_c \tag{24}$$

$$\gamma_\varphi(\kappa) = \kappa \frac{d}{d\kappa} \ln Z_\varphi \Big|_c \tag{25}$$

and the derivative is taken along an appropriately chosen curve in the phase diagram, which we here denote by c. Similarly, integration of the renormalization group equation for any multiplicatively renormalizable $\Gamma_t^{(N,M)}$ yields

$$\Gamma_t^{(N,M)}(t, \lambda, \varphi) = \exp\left(\int_\kappa^{m_t} \left(\frac{N}{2}\gamma_\varphi - M\gamma_{\varphi^2}\right) \frac{dx}{x}\right) \Gamma_t^{(N,M)}(t(\kappa), \lambda(\kappa), \varphi(\kappa)). \tag{26}$$

To impose a specific, as opposed to abstract, coordinate transformation between bare and renormalized theory the renormalization constants Z_φ, Z_{φ^2} and Z_λ must be fixed. Here, we impose the explicitly magnetization-dependent normalization conditions

$$\partial_{p^2}\Gamma_t^{(2)}(p, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa)\big|_{p^2=0} = 1 \tag{27}$$

$$\Gamma_t^{(2,1)}(0, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa) = 1 \tag{28}$$

$$\Gamma_t^{(4)}(0, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa) = \lambda. \tag{29}$$

Note that in this case we impose the normalization conditions on the transverse correlation functions. These conditions serve to fix the three Z functions associated with φ_B, t_B and λ_B while the condition

$$\kappa^2 = \Gamma_t^{(2)}(0, t(\kappa, \kappa), \lambda(\kappa), \varphi(\kappa), \kappa) \tag{30}$$

serves as a gauge fixing condition that relates the sliding renormalization scale κ to the physical temperature t and the physical magnetization φ. Physically, κ is a fiducial value of the nonlinear scaling field m_t, which is the inverse transverse correlation length.

Besides m_t, the other nonlinear scaling field we use to parameterize our results is

$$m_\varphi^2 = \frac{1}{3} \frac{\Gamma_t^{(4)} \varphi^2}{\partial_{p^2}\Gamma_t^{(2)}\big|_{p^2=0}} \tag{31}$$

which is an RG invariant. It represents the anisotropy in the masses of the longitudinal and transverse modes and is related to the stiffness constant $\rho_s = \varphi^2 \partial_{p^2}\Gamma_t^{(2)}\big|_{p^2=0}$ via $m_\varphi^2 = \frac{1}{3}\lambda\rho_s$.

With a given renormalization prescription one may determine the equation of state in terms of the nonlinear scaling fields m_t and m_φ, as the transverse and longitudinal propagators that appear in all perturbative diagrams can be parameterized in terms of them. One of the main motivations for this reparameterization in terms of nonlinear scaling fields is that it eliminates all tadpole diagrams at higher loop order. However, what is required is the equation of state in terms of the linear scaling fields t and φ. One must therefore determine the coordinate transformation, $t = t(m_t, m_\varphi)$, $\varphi = \varphi(m_t, m_\varphi)$, between them.

3.2. Relating nonlinear and linear scaling fields

This may be done by specifying a particular curve, c, in the phase diagram along which we integrate the differential relation for the transverse vertex functions $\Gamma_t^{(N)}$. For example,

$$\mathrm{d}\Gamma_t^{(2)} = \Gamma_t^{(2,1)}\,\mathrm{d}t + \tfrac{1}{6}\Gamma_t^{(4)}\,\mathrm{d}\varphi^2 \tag{32}$$

can be integrated along a curve of constant φ to yield $\mathrm{d}t = \mathrm{d}\Gamma_t^{(2)}\big/\Gamma_t^{(2,1)}$, where the right-hand side is naturally written in the coordinate system (m_t, m_φ). To integrate the renormalization group equation for the vertex functions it is most natural to use $m_t = \kappa$ as the flow variable and hold m_φ constant. However, if we then wish to integrate (32) along a curve of constant φ we must include a Jacobian factor, $2/(2 - \gamma_\lambda + \gamma_\varphi)$, that takes into account the variation of m_φ along a constant φ curve. The relation between m_φ and φ is specified by (31) using the renormalization group equations for $\partial_{p^2}\Gamma_t^{(2)}$ and $\Gamma_t^{(4)}$ with the normalization conditions (27) and (29). Using (26) for $\Gamma_t^{(2)}$ and $\Gamma_t^{(2,1)}$ and the normalization conditions (28) and (30) one may write

$$\frac{\mathrm{d}\Gamma_t^{(2)}}{\Gamma_t^{(2,1)}} = (2 - \gamma_\varphi) \exp\left(-\int_{\kappa_0}^{\kappa} \gamma_{\varphi^2}\frac{\mathrm{d}x}{x}\right) \kappa\,\mathrm{d}\kappa. \tag{33}$$

In the universal limit, where $\lambda \to \infty$ (this limit can also be accessed by fixing the coupling at its asymptotic fixed point value), the crossover to mean-field theory is pushed off to infinity and the theory is then controlled in the limit $\kappa \to \infty$ by the Wilson–Fisher fixed point. Hence,

$\gamma_i \to \gamma_i^{\mathrm{WF}}$, where γ_i^{WF} is the Wilson function γ_i at the Wilson–Fisher fixed point. Defining $\Delta\gamma_i = \left(\gamma_i - \gamma_i^{\mathrm{WF}}\right)$ (33) yields

$$\mathrm{d}t = (2 - \gamma_\varphi)\exp\left(-\int_{\kappa_0}^{\kappa} \Delta\gamma_{\varphi^2}\frac{\mathrm{d}x}{x}\right)\left(\frac{\kappa}{\kappa_0}\right)^{-\gamma_{\varphi^2}}\kappa\,\mathrm{d}\kappa. \tag{34}$$

To integrate (34) we need to fix some boundary condition. The coexistence curve, where $m_t = 0$, is a natural one. In this case, $\mathrm{d}t$ integrates to the temperature variable $(T - T_c(\varphi))$, where $T_c(\varphi)$ corresponds to that point on the coexistence curve where the magnetization is φ. As we wish to use as temperature variable $t = (T - T_c)$, we may write $(T - T_c(\varphi)) = t + \Delta$, where $\Delta = (T_c - T_c(\varphi))$ is the temperature shift that measures the distance between the critical point and the point on the coexistence curve, $T_c(\varphi)$. In the integral, $-\int_{\kappa_0}^{\kappa}\Delta\gamma_{\varphi^2}\,\mathrm{d}x/x$, one may safely take the universal limit, $\kappa_0 \to \infty$. In this universal scaling limit, in terms of the linear scaling fields t and φ, the problem has only one scaling variable, x. In terms of the coordinates m_t and m_φ, this manifests itself as a reduction to the single scaling variable, $z = m_t/m_\varphi$. Passing to this variable, using (31), and integrating this along a curve of constant φ (taking into account the Jacobian factor) one finds

$$A_1(1 + x) = \mathcal{F}(z) \tag{35}$$

where the scaling variable $x = B^{1/\beta}t/\varphi^{1/\beta}$, B being the non-universal amplitude introduced previously, and the universal scaling function $\mathcal{F}(z)$ is

$$\mathcal{F}(z) = \int_0^z \frac{2(2 - \gamma_\varphi)}{2 - \gamma_\lambda + \gamma_\varphi}D(x)x^{\frac{1}{\beta}}\frac{\mathrm{d}x}{x},$$

where

$$D(x) = \exp\left(-\int_\infty^x 2\left(\frac{\Delta\gamma_{\varphi^2} - \frac{\Delta\gamma_\lambda}{2\beta} + \frac{\Delta\gamma_\varphi}{2\beta}}{2 - \gamma_\lambda + \gamma_\varphi}\right)\frac{\mathrm{d}y}{y}\right). \tag{36}$$

The quantity A_1 is related to the universal zero of the equation of state. In terms of the amplitude B,

$$B^2 = \frac{\lambda}{3\kappa^{(4-d-\eta)}}A_1^{2\beta}. \tag{37}$$

Equation (35) determines the coordinate transformation, $t = t(m_t, m_\varphi)\ \varphi = \varphi(m_t, m_\varphi)$, in the scaling limit where there is only one relevant scaling variable associated with the linear scaling fields, $x \sim t/\varphi^{1/\beta}$, and one relevant scaling variable associated with the nonlinear scaling fields, $z = m_t/m_\varphi$. Hence, we determine the coordinate transformation $x = x(z)$. Note that the geometry of this transformation has some unusual properties relative to the (T, φ) plane. The coexistence curve is mapped to the single point $x = -1$, the critical isotherm is mapped to the single point $x = 0$, and the critical isochor, for $t > 0$, maps to the single point at infinity. However, the critical point itself—as the intersection of the coexistence curve, critical isotherm and critical isochor—maps to all these points.

To proceed to the equation of state we use (16) and the renormalization group equation for $\Gamma_t^{(2)}$. Once again, passing to the variable z and including in the Jacobian factor, one obtains

$$\frac{H}{\varphi^\delta} = \left(\frac{\lambda}{3\kappa^2}\right)^{\frac{(\delta-1)}{2}}\mathcal{G}(z), \tag{38}$$

where the universal scaling function $\mathcal{G}(z)$ is given by

$$\mathcal{G}(z) = z^{\frac{\gamma}{\beta}}\exp\left(\frac{\gamma}{\beta}\int_\infty^z \frac{\Delta\gamma_\lambda - \Delta\gamma_\varphi}{2 - \gamma_\lambda + \gamma_\varphi}\frac{\mathrm{d}y}{y}\right)\exp\left(-\int_\infty^z \frac{2\Delta\gamma_\varphi}{2 - \gamma_\lambda + \gamma_\varphi}\frac{\mathrm{d}y}{y}\right). \tag{39}$$

Introducing the scaling variable $y = A_3^{-1}(H/\varphi^\delta)$, where A_3 is related to the non-universal amplitude B_c via

$$B_c^\delta = A_3 \left(\frac{\lambda}{3\kappa^2} \right)^{\frac{(\delta-1)}{2}}. \tag{40}$$

One sees that the universal equation of state is now of the form (1) with

$$f(x) = \frac{1}{A_3} \mathcal{G}(\mathcal{F}^{-1}(A_1(1+x))). \tag{41}$$

We may now ask what more can be said about A_1 and A_3, or whether they are simply non-universal parameters, being related to the non-universal amplitudes B_c and B, that cannot be determined within the present formalism? A_1, in fact, is related to the universal zero of the equation of state, and may be calculated in the following manner: we choose some arbitrary value of z, z_0, and write $\mathcal{F}(z) \rightarrow \mathcal{F}(z_0) + \int_{z_0}^z F(x)\,\mathrm{d}x$; then choose z and z_0 in the asymptotic regime $z, z_0 \rightarrow \infty$ $z > z_0$, wherein, from Griffith's analyticity, we can write $\mathcal{F}(z) = z^{1/\beta} \sum_{n=0}^\infty \mathcal{F}_n^\infty z^{-2n}$. In the limit $z \rightarrow \infty$, depending on the value of β, certain terms in the asymptotic expansion diverge. For instance, for the three-dimensional Ising model, at one loop, $\beta = 3/10$, hence, only the $n = 0$ and $n = 1$ terms diverge, while the contributions from $n \geqslant 2 \rightarrow 0$. Denoting the divergent part of the expansion as $\mathcal{F}^\infty(z)$; as z_0 is constant we may identify the non-constant term, A_1x, in this limit with $\mathcal{F}^\infty(z)$. Hence, we may identify $A_1 = \lim_{z_0 \rightarrow \infty}(\mathcal{F}(z_0) - \mathcal{F}^\infty(z_0))$. In terms of the integrand, $I(x)$, of the scaling function $\mathcal{F}$, we may write

$$A_1 = \int_0^\infty (I(x) - I^\infty(x))\,\mathrm{d}x, \tag{42}$$

where $I^\infty(x)$ is defined via $\mathcal{F}^\infty = \int_0^z I^\infty(x)\,\mathrm{d}x$. A_1 is clearly universal. To determine A_3, we set the condition $y = 1$ on the critical isotherm $t = 0$. This corresponds to a particular value, z_c, of z. Hence, $A_3 = \mathcal{G}(z_c)$. To determine z_c note that from (36) on the critical isotherm $A_1 = \mathcal{F}(z_c)$. The inversion of this function allows for the identification of z_c. Hence, we deduce A_3 to be

$$A_3 = \mathcal{G}(\mathcal{F}^{-1}(A_1)) \tag{43}$$

which is, once again, a universal function.

So, given that A_3 is determined from A_1, and A_1 is determined from $\mathcal{F}$ by subtracting off its divergent component as $z \rightarrow \infty$, we see that the simple ingredients that enter into a complete specification of the universal equation of state are the three Wilson functions— γ_λ, γ_{φ^2} and γ_φ. These are the only quantities that need to be determined perturbatively (or otherwise).

Note that this equation of state has been determined from a first principles calculation based on an underlying microscopic model. It is parameterized, but parameterized in a way that is completely determined by the underlying model. This is in distinction to standard parametric representations [1, 11], where, after imposing certain analyticity requirements, there is a large arbitrariness in determining the scaling functions $h(\theta)$ and $m(\theta)$. In fact, these functions may depend on an arbitrary number of parameters and, for each parameter, a universal quantity must be independently calculated in order to fix it. Also, the parameter z here has a much more transparent and direct physical meaning than θ, being simply the ratio of the two fundamental nonlinear scaling fields in the problem—m_t and m_φ—the transverse correlation length and the stiffness constant. As these quantities are well defined throughout the phase diagram this formulation has an added advantage relative to parametric, fixed-dimension expansions where the relevant nonlinear scaling field used is the mass for $T > T_c$ and there are difficulties reaching the ordered phase [1].

In order to determine the expansion coefficients f_n^0 and f_n^∞, as introduced in section 1, one requires the Taylor expansion of $f(x)$ around $x = 0$ and $x = \infty$. In terms of our parametric representation, $\mathrm{d}^n f(x)/\mathrm{d}x^n$ can be expressed using $\mathrm{d}/\mathrm{d}x = (\mathrm{d}z/\mathrm{d}x)\,\mathrm{d}/\mathrm{d}z$, where $\mathrm{d}z/\mathrm{d}x = A_1/(\mathrm{d}\mathcal{F}(z)/\mathrm{d}z)$ hence,

$$\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} = \frac{A_1}{A_3}\left(\left(\frac{\mathrm{d}\mathcal{F}(z)}{\mathrm{d}z}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \mathcal{G}(z) \tag{44}$$

which need to be evaluated at the points of interest $z = z_c$ $(x = 0)$, $z = \infty$ $(x = \infty)$ and $z = 0$ $(x = -1)$. For instance, taking the limit $x \to \infty$ in (36) and (39), and using the fact that the Wilson functions γ_i approach their values at the Wilson–Fisher fixed point so $\Delta\gamma_i \to 0$, one finds

$$\mathcal{F}(z) \to \gamma z^{1/\beta} \tag{45}$$

$$\mathcal{G}(z) \to z^{\frac{\gamma}{\beta}}. \tag{46}$$

Hence,

$$f(x) \to \frac{A_1^\gamma}{A_3}\gamma^{-\gamma}x^\gamma + O(x^{\gamma-2\beta}) \tag{47}$$

from which, using (43) we may identify the expansion coefficient

$$f_0^\infty = \frac{(A_1/\gamma)^\gamma}{\mathcal{G}(\mathcal{F}^{-1}(A_1))} \tag{48}$$

which is related to the universal amplitude ratio R_χ via $f_0^\infty = R_\chi^{-1}$. Using the expression for C^+ in our formulation, $C^+ = \kappa^{-(2-\eta)}\gamma^{-\gamma}$ and equations (37) and (40), one can verify that (48) is identical to the expression $R_\chi = (C^+/B_c)(B/B_c)^{\delta-1}$. Similarly, using the expression for C_4^+ in our formulation

$$C_4^+ = -\lambda\gamma^{2\gamma+\mathrm{d}\nu}\kappa^{4\gamma-8+2\mathrm{d}\nu}. \tag{49}$$

One may determine the universal amplitude ratio $R_4^+ = -C_4^+ B^2 \big/ (C^+)^3$ to be

$$R_4^+ = \frac{3(\mathcal{F}(z_c))^{2\beta}}{\gamma^{\gamma-\mathrm{d}\nu}}. \tag{50}$$

With these two amplitude ratios in hand two-scale factor universality implies that any other may be determined.

4. One-loop results

The advantages of the present formulation can be best illustrated by considering a concrete example. We will consider the universal equation of state in the one-loop approximation, as at this level it is still possible to obtain analytic or 'quasi'-analytic results.

We begin with the values of the Wilson functions to one loop. The running dimensionless coupling λ satisfies

$$z\frac{\mathrm{d}\lambda(z)}{\mathrm{d}z} = -\varepsilon\lambda + c_d\lambda^2(z)\left(\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{9}\right), \tag{51}$$

where $c_d = 3(4-d)\Gamma((4-d)/2)/2(4\pi)^{d/2}$. Taking the initial condition $\lambda(z_0) = \lambda$, in the limit $z_0 \to \infty$, $\lambda \to \infty$ one arrives at the universal separatrix solution

$$\lambda(z) = \left(c_d\left(\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{9}\right)\right)^{-1}; \tag{52}$$

this solution may also be reached by choosing the initial coupling to be on the separatrix solution at $z = z_0$. On the separatrix

$$\gamma_\lambda = (4 - d) \left(\frac{\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{9}}{\left(1 + \frac{1}{z^2}\right)^{\frac{d-4}{2}} + \frac{(N-1)}{9}} \right) \tag{53}$$

$$\gamma_{\varphi^2} = (4 - d) \left(\frac{\left(1 + \frac{1}{z^2}\right)^{\frac{d-6}{2}} + \frac{(N-1)}{3}}{3\left(1 + \frac{1}{z^2}\right)^{\frac{d-4}{2}} + \frac{(N-1)}{3}} \right) \tag{54}$$

$$\gamma_\varphi = 0. \tag{55}$$

In the limit $z \to \infty$, the Wilson–Fisher fixed point is approached and $\gamma_i \to \gamma_i^{WF}$ with, at one loop, $\gamma_\lambda = (4 - d)$ and $\gamma_{\varphi^2} = (4 - d)(N + 2)/(N + 8)$. In contrast, in the limit $z \to 0$ the strong-coupling fixed point is approached and $\gamma_i \to \gamma_i^{SC}$. For $N > 1$ the Goldstone bosons dominate and $\gamma_\lambda = \gamma_{\varphi^2} = (4 - d)$. For $N = 1$ however, this fixed point is mean field like as fluctuations are suppressed and $\gamma_i \to 0$.

In the limit $z \to \infty$, the Wilson functions can be expanded as power series in z^{-2} for any N

$$\gamma_i(z) = \gamma_i^{WF} + \sum_{n=1}^{\infty} a_i(n) z^{-2n}. \tag{56}$$

Hence, the universal scaling functions $\mathcal{F}$ and $\mathcal{G}$ can also be written as power series in z^{-2}. This is true in a diagrammatic expansion to all orders not just at one loop. The limit $z \to 0$ is more complicated. In this limit, $\gamma_i \to \gamma_i^{SC}$ but the nature of the corrections is not obvious. At the one-loop level, from (53), one can see that the leading corrections to the strong-coupling fixed point values will be $z^{(4-d)/2}$.

For $N \neq 1$ analytic progress is difficult. However, for $N = 1$ these expressions simplify greatly yielding

$$\gamma_\lambda = (4 - d) \left(1 + \frac{1}{z^2}\right)^{-1} \tag{57}$$

$$\gamma_{\varphi^2} = \frac{(4 - d)}{3} \left(1 + \frac{1}{z^2}\right)^{-1}. \tag{58}$$

With these expressions one can explicitly calculate the scaling functions $\mathcal{G}(z)$ and $\mathcal{F}(z)$:

$$\mathcal{G}(z) = z^2 \left(\frac{2}{(d - 2)} + z^2 \right)^{\frac{(4-d)}{(d-2)}} \tag{59}$$

$$\mathcal{F}(z) = \frac{3}{d + 2} \left(\left(\frac{2}{d - 2} + z^2 \right)^{\frac{2(4-d)}{3(d-2)}} (2z^2 - 1) + \left(\frac{2}{d - 2} \right)^{\frac{2(4-d)}{3(d-2)}} \right). \tag{60}$$

To determine the constant A_1 we take the $z \to \infty$ limit of (60), identify the divergent part with $A_1 x$ and the constant remainder with A_1.[5] Note that the case $d = 2$ is problematic. This

---

[5] If $2(4 - d)/3(d - 2)$ is a positive integer, n, then this remainder is zero and A_1 cannot be determined by looking at the asymptotic limit $z \to \infty$. This is a pure artefact of the one-loop approximation, where $\eta = 0$, and has no physical meaning.

is an artefact of the one-loop approximation where, in particular, in $d = 2$ one has $\beta = 0$, which is clearly unphysical. The case $d = 4$ is also inaccessible due to the fact that we took the universal limit $\lambda \to \infty$. It may be recovered by returning to (51) and integrating it and not taking this limit. In this case, as expected, logarithms appear. So, equations (59) and (60) are valid for $2 < d < 4$. A_3 can be determined as a function of the undetermined A_1 using (43). Explicitly, taking the $z \to \infty$ limit A_1 can be identified to be

$$A_1 = \frac{3}{d+2}\left(\frac{2}{d-2}\right)^{\frac{2(4-d)}{3(d-2)}} \tag{61}$$

from which one determines $z_c = 2^{-1/2}$ which, interestingly, is dimension independent, though, once again, this is a one-loop artefact. With z_c in hand one determines A_3 to be

$$A_3 = \frac{1}{2}\left(\frac{d+2}{2(d-2)}\right)^{\frac{(4-d)}{(d-2)}}. \tag{62}$$

Thus, the one-loop equation of state for $N = 1$ in d dimensions is

$$y = \frac{2^{\frac{2}{(d-2)}}}{(d+2)^{\frac{(4-d)}{(d-2)}}}z^2(2 + (d-2)z^2)^{\frac{(4-d)}{(d-2)}} \tag{63}$$

$$x = (2z^2 - 1)(1 + (d-2)z^2/2)^{\frac{2(4-d)}{3(d-2)}}. \tag{64}$$

Griffiths analyticity can be seen quite simply from these expressions. For $x \to \infty$ one has $z \to \infty$ and the right-hand side of (64) takes the form $z^{1/\beta}\sum_{n=0}^{\infty}(c_n/z^{2n})$, where $\beta = 3(d-2)/2(d+2)$. Substituting into (63), one then has the expansion (3), where $\gamma = 6/(d+2)$. Similarly, by expanding (64) around $z = z_c$ one obtains the expansion (2). Finally, in the vicinity of the coexistence curve, from (64) we see that $(1+x)$ can be written as a power series expansion in z^2 and, hence, we verify (11). Once again, within the approximation used one would not expect essential singularities to appear. To summarize: (63) and (64) have been determined from first principles from the underlying microscopic theory and obey all analyticity and other properties required in the different asymptotic regimes.

From these expressions one may calculate analytically the coefficients $f_n^0(d)$ and $f_n^{\infty}(d)$ as functions of d, as well as derived quantities, such as the r_{2n}, and universal amplitude ratios, like R_χ and R_4^+. In the same way, we may examine the behaviour in the vicinity of the coexistence curve. First, we determine z as a power series in $(1 + x)$, and then we replace the result in the expansion of y for z small.

In table 1 we show the coefficients $f_1^0(d) - f_5^0(d)$ and in table 2 $f_1^c(d) - f_3^c(d)$. The coefficients $f_0^{\infty}(d) - f_5^{\infty}(d)$ for arbitrary d appear in table 3. Our results for some important quantities like $r_{2n}(d)$ appear in table 4. Obviously with analytic expressions in hand it is straightforward to generate other coefficients.

One may also recover the well-known ε-expansion result by substituting $d = 4 - \varepsilon$ into (63) and (64), obtaining

$$y = \frac{2^{\frac{2}{(2-\varepsilon)}}}{(6-\varepsilon)^{\frac{\varepsilon}{(2-\varepsilon)}}}z^2(2(1+z^2) - \varepsilon z^2)^{\frac{\varepsilon}{(2-\varepsilon)}} \tag{65}$$

$$x = (2z^2 - 1)(1 + (2-\varepsilon)z^2/2)^{\frac{2\varepsilon}{3(2-\varepsilon)}}. \tag{66}$$

Expanding in powers of ε to $O(\varepsilon)$ one finds

$$y = 2z^2\left(1 + \frac{\varepsilon}{2}\ln\frac{2(1+z^2)}{3}\right) \tag{67}$$

Table 1. Values of f_n^0 for the d-dimensional Ising universality class.

Coefficient	Value
f_1^0	$3 \times 2^{-\frac{1}{3}+\frac{8}{3(d-2)}}(2+d)^{\frac{2+d}{6-3d}}$
f_2^0	$-(d-4)(2+d)^{\frac{4+2d}{6-3d}}2^{\frac{8(4-d)}{3(d-2)}}$
f_3^0	$(1/3) \times 2^{-3+\frac{8}{d-2}}(d-4)(2+d)^{\frac{2+d}{2-d}}$
f_4^0	$(1/27) \times 2^{\frac{70-19d}{3d-6}}(d-7)(d-4)(2+d)^{\frac{8+4d}{6-3d}}(8+d)$
f_5^0	$\dfrac{(d-16)(d-4)(5+d)(2d-11)}{405 \times 2^{\frac{20(d-4)}{3(d-2)}}(2+d)^{\frac{5(2+d)}{3(d-2)}}}$

Table 2. Values of f_n^c for the d-dimensional Ising universality class.

Coefficient	Value
f_1^c	$3 \times 2^{\frac{6-d}{d-2}}(2+d)^{\frac{2}{2-d}}$
f_2^c	$3 \times 2^{\frac{10-4d}{d-2}}\left(-3 \times 2^{\frac{2d-2}{d-2}}+5 \times 2^{\frac{d}{d-2}}\right)\frac{d-4}{(2+d)^2}(2+d)^{\frac{4-d}{d-2}}$
f_3^c	$64^{\frac{d-1}{2-d}}\frac{4-d}{(2+d)^3}(2+d)^{\frac{4-d}{d-2}}\Big(81 \times 2^{\frac{3d+4}{d-2}}+47 \times 2^{\frac{2+4d}{d-2}}$
	$\qquad -45 \times 32^{\frac{d}{d-2}}-35 \times 2^{\frac{2(3+d)}{d-2}}d+9 \times 2^{\frac{2+4d}{d-2}}d\Big)$

Table 3. Values of f_n^∞ for the d-dimensional Ising universality class.

Coefficient	Value
f_0^∞	$\dfrac{4^{\frac{1}{d-2}}\left(\frac{2+d}{d-2}\right)^{\frac{d-4}{d-2}}}{\left(2^{\frac{14-5d}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{6}{2+d}}}$
f_1^∞	$\dfrac{4^{\frac{1}{d-2}}}{d-2}\left(2^{\frac{14-5d}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{3(d-4)}{2+d}}\left(\frac{2+d}{d-2}\right)^{\frac{d-4}{d-2}}$
f_2^∞	$-3 \times 2^{\frac{6-2d}{d-2}}\left(\frac{d-4}{d-2}\right)\left(2^{\frac{14-5d}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{6(d-3)}{2+d}}\left(\frac{2+d}{d-2}\right)^{\frac{d-4}{d-2}}$
f_3^∞	$2^{\frac{2(3-d)}{d-2}}\dfrac{(d-4)(4d-13)}{(d-2)^3}\left(2^{\frac{14-5d}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{9d-24}{2+d}}\left(\frac{2+d}{d-2}\right)^{\frac{d-4}{d-2}}$
f_4^∞	$2^{\frac{12-5d}{d-2}}\dfrac{d-4}{(d-2)^4}\left(2^{\frac{14-5d}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{12d-30}{2+d}}\left(\frac{2+d}{d-2}\right)^{\frac{d-4}{d-2}}$
	$\qquad \times (17-5d)(11d-32)$
f_5^∞	$\dfrac{2^{\frac{2(8d-35)}{d-2}}(d-2)^{\frac{50-15d}{d-2}}\left(\frac{2+d}{d-2}\right)^{\frac{d-4}{d-2}}}{-45\left(2^{\frac{14-5d}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{66}{2+d}}} \times \Big(176\,553 \times 2^{\frac{6d+8}{d-2}}d$
	$\qquad +27(2419d-11973)d^2 2^{\frac{4(3+d)}{d-2}}-189(1520+13d^4)2^{\frac{5(2+d)}{d-2}}\Big)$

$$1+x = 2z^2 + \frac{\varepsilon}{3}(2z^2-1)\ln(1+z^2). \tag{68}$$

Inverting (68) in powers of ε and substituting into (67) one finds

$$y = 1+x+\frac{\varepsilon}{2}(1+x)\ln\frac{(x+3)}{3}-\frac{\varepsilon}{3}x\ln\frac{(x+3)}{2} \tag{69}$$

which is the well-known result in terms of normalized variables.

Table 4. Values of R_4^+, F_0^∞ and some r_{2n} coefficients for the d-dimensional Ising universality class.

Coefficient	Value
R_4^+	$\frac{6}{d-2}\left(2^{\frac{5}{3}+\frac{4}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{3(d-2)}{2+d}}$
F_0^∞	$\dfrac{4^{\frac{d-1}{2-d}}(d-2)^2\left(\frac{2+d}{d-2}\right)^{\frac{4-d}{d-2}}}{9\left(2^{\frac{14-5d}{6-3d}}(d-2)^{\frac{8-2d}{3d-6}}\right)^{\frac{6(d-3)}{2+d}}}$
r_6	$5/2 \times (4-d)(d-2)$
r_8	$35/6 \times (d-4)(4d-13)$
r_{10}	$35/4 \times (4-d)(5d-17)(11d-32)$
r_{12}	$385 \times 2^{\frac{2-6d}{d-2}}\left(665 \times 2^{4+\frac{5d}{d-2}} - 6539 \times 2^{6+\frac{10}{d-2}}d\right.$
	$\left. + 11973 \times 2^{\frac{2+4d}{d-2}}d^2 - 2419 \times 2^{\frac{2+4d}{d-2}}d^3 + 91 \times 32^{\frac{d}{d-2}}d^4\right)$

4.1. The case $d = 3$

For $d = 3$ things are even more transparent, as the parameter z can be totally eliminated, thereby ending with a direct, unparameterized relation between x and H/φ^δ. For $d = 3$ the scaling functions $\mathcal{F}(z)$ and $\mathcal{G}(z)$ are

$$\mathcal{F}(z) = \frac{2^{\frac{2}{3}}3}{5} + \frac{3}{5}(z^2+2)^{\frac{2}{3}}(2z^2-1) \tag{70}$$

$$\mathcal{G}(z) = (z^4 + 2z^2) \tag{71}$$

while A_1 and A_3 are given by

$$A_1 = \frac{2^{\frac{2}{3}}3}{5} \qquad A_3 = \frac{5}{4}. \tag{72}$$

With these values for the amplitudes A_1 and A_3 the equation of state in terms of the variable z is

$$z^4 + 2z^2 - \frac{5y}{4} = 0 \tag{73}$$

which can be simply solved (the positive square root is required) and substituted into (70) to find

$$2^{\frac{2}{3}}x = \left(1 + \left(1 + \frac{5y}{4}\right)^{\frac{1}{2}}\right)^{\frac{2}{3}}\left(2\left(1 + \frac{5y}{4}\right)^{\frac{1}{2}} - 3\right) \tag{74}$$

which can be seen to satisfy $y = 1$ at $x = 0$ and $y = 0$ at $x = -1$. For large x, $y \to (2^{8/5}/5)x^{6/5}$, i.e. $y \sim x^\gamma$ as required by Griffith's analyticity. It is valid for both $t > 0$ and $t < 0$.

In the second column of table 5 we see the numerical values of the coefficients f_n^0 and f_n^∞, as well as some important derived quantities, such as the r_{2n}. All these values are as in good an agreement with known values as one might expect from a one-loop calculation, in those cases where a comparison can be made, with one apparent exception: the value of r_8 is about 2–3 times bigger than the majority of estimates, which are in the range 2.18–2.7. However, a Monte Carlo simulation of Kim and Landau [10] led to a value of r_8 which was
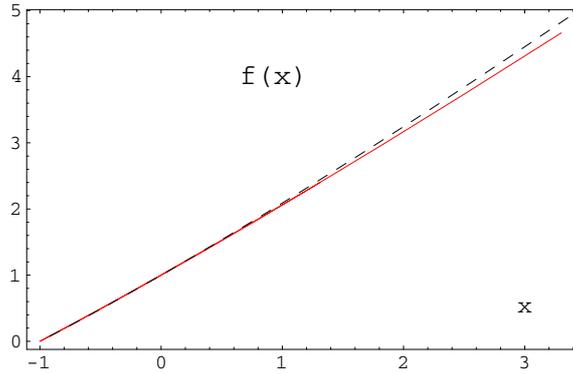
Figure 1. The scaling function $f(x)$ of the three-dimensional Ising model. The dashed line was taken from [8].

much larger than other estimates. However, this estimate goes down when we fit with the exact exponents in the next section. Our value of $r_{10} = -17.5$ is in the expected region of previous calculations, which give estimates in the region (-4)–(-25), though the errors associated with many of these estimates are large. Our estimate of r_{12} for $d = 3$ would seem to be new. Obviously an important advantage of the present approach is the facility with which the r_{2n} can be calculated for even very high n. In figure 1 we see a comparison between our one-loop equation of state and that obtained by the HT method.

5. Fitted exponents

Although one of the chief advantages of the present methodology is the fact that it is completely self-contained, in that there are no parameters to be fixed by appealing to exogenous information, as in standard parametric approaches, it is possible to adapt the present method to utilize information that is available, such as precise estimates for the critical exponents and amplitude ratios. To illustrate this here, we once again consider the case $N = 1$. We take the one-loop 'crossover' function $(1 + z^{-2})$ to give the exact form of the crossover and fit its asymptotic value to the best estimates for the critical exponents. Hence, we take

$$\gamma_\lambda = \frac{(4-d)}{\left(1 + \frac{1}{z^2}\right)} \qquad \gamma_{\varphi^2} = \frac{(2 - 1/\nu)}{\left(1 + \frac{1}{z^2}\right)} \qquad \gamma_\varphi = \frac{\eta}{\left(1 + \frac{1}{z^2}\right)}. \tag{75}$$

The crossover form for η is of course a pure supposition given that the form $(1+z^{-2})$ is derived from a one-loop calculation. The form of (75) is such as to guarantee a crossover to the known asymptotic behaviour. The same procedure could be carried out using a two-loop calculation. In this case the crossover functions for each Wilson function would be different, due to the fact that different diagrams contribute to them. Once again constants would be introduced to ensure a crossover in the limit $z \to \infty$ to the correct exponent values for ν and η. With the ansatz (75) one finds

$$\mathcal{F}(z) = \left(\frac{\nu}{\beta}\right)^{\frac{1}{2\beta}} \frac{2\beta(\gamma - 1)}{1 - 2\beta} \left[\left(\frac{\gamma(1 - 2\beta)}{2\nu(\gamma - 1)} z^2 - 1\right) \left(1 + \frac{\beta}{\nu} z^2\right)^{\frac{1}{2\beta} - 1} + 1 \right] \tag{76}$$

$$\mathcal{G}(z) = z^{\frac{\gamma}{\beta}} \left(1 + \frac{\nu}{\beta z^2}\right)^{\frac{(\gamma - 2\beta)}{2\beta}}. \tag{77}$$

Table 5. Numerical values of expansion coefficients for the three-dimensional Ising universality class: HT results taken from [11], one loop (RG1) and fitted exponents (RGA) results. The resulting values for the coefficients r_{2n} were obtained from relation (8) and using the values of f_n^∞.

	HT	RG1	RGA
f_0^∞	0.6024(15)	0.606	0.596
f_1^∞		0.696	0.793
f_2^∞		0.6	0.613
f_3^∞		0.223	0.151
f_4^∞		−0.066	−0.084
f_5^∞		−0.0454	0.005 27
r_6	2.056(5)	2.5	1.938
r_8	2.3(1)	5.833	2.505
r_{10}	−13(4)	−17.5	−12.599
r_{12}		−192.5	10.902
f_1^0	1.0527(7)	1.034	1.05
f_2^0	0.0446(4)	0.029	0.043
f_3^0	−0.0254(7)	−0.0034	−0.0054
f_4^0		0.0007	0.0013
f_5^0		−0.000 19	−0.000 4
f_1^c	0.9357(11)	0.96	0.939
f_2^c	0.08(7)	0.048	0.076
f_3^c		−0.0112	−0.023
f_4^c		0.0049	0.0138
f_5^c		−0.0028	−0.011
R_4^+	7.81(2)	6.892	7.981
F_0^∞	0.033 82(15)	0.0347	0.0263

Examining the large z limit one determines the universal amplitude A_1 to be

$$A_1 = \left(\frac{\nu}{\beta}\right)^{\frac{1}{2\beta}} \frac{2\beta(\gamma - 1)}{1 - 2\beta}. \tag{78}$$

Consequently, we determine $z_c^2 = 2\nu(\gamma - 1)/\gamma(1 - 2\beta)$ and

$$A_3 = \frac{2\beta(\gamma - 1)}{(\gamma - 2\beta)} \left(\frac{2\nu(\gamma - 2\beta)}{2\gamma\beta(1 - 2\beta)}\right)^{\frac{\gamma}{2\beta}}. \tag{79}$$

Using this mechanism, in the third column of table 5, we show the results for the three-dimensional Ising model. The values of the critical exponents that we have used are the best values reported in the literature [11]. i.e. $\gamma = 1.2372$, $\beta = 0.3265$ and $\nu = 0.6301$.

Most of the values found with fitted exponents substantially improve the value compared with the one-loop approximation in those cases where a comparison can be made with HT expansions. This is particularly notable in r_6, r_8 and r_{10}. f_0^∞ and F_0^∞ are a little puzzling in this respect. However, it is worth noting the experimental result reported in [11] where $f_0^\infty = 0.5917$, this being substantially different from the theoretical value of $f_0^\infty = 0.6369$ [11]. The only other notable change is that of f_5^∞, which is related to r_{12}, where there is a sign change passing from one loop to the adjusted values. In figure 2 we see a comparison between our three-dimensional Ising model using fitted exponents and the result using high precision calculations.

In the same way, we can also substitute the well-known exact values for the critical exponents in the two-dimensional case, i.e. $\gamma = 7/4$, $\beta = 1/8$ and $\nu = 1$. Table 6 shows the
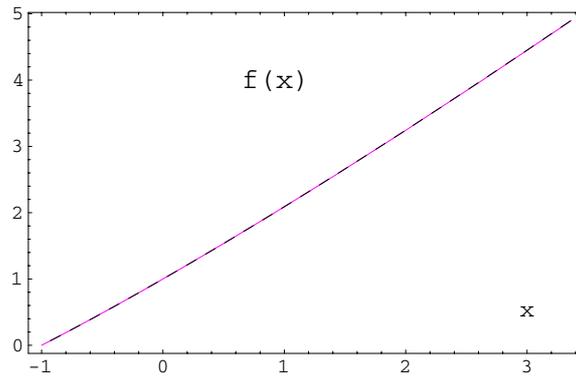
Figure 2. The scaling function $f(x)$ of the three-dimensional Ising model using fitted exponents. The dashed line was taken from [8]. Differences are not visible at this scale.

Table 6. Numerical values of expansion coefficients for the two-dimensional Ising universality class; high precision (HT) results taken from [11] and results from using fitted exponents (RGA).

	HT	RGA
f_0^∞	0.14753	0.1043
f_1^∞		0.1696
f_2^∞		0.2365
f_3^∞		0.2931
r_6	3.678 67(7)	2.8571
r_8	26.041(11)	15.2381
r_{10}	284.5(2.4)	125.714
r_{12}	$4.2(7) \times 10^3$	1436.73
f_1^0		1.1724
f_2^0		0.1473
f_3^0		-0.0164
f_1^c		0.785
f_2^c		0.295
R_4^+	7.336 774(10)	9.7594
F_0^∞	$5.923 57(6) \times 10^{-5}$	0.100 67

subsequent results. In this case, except for the value of f_0^∞, the values found for r_{2n} show significant differences. This probably hints at the inadequacy of the ansatz for the crossover functions (75). We have not been able to find reported values for the coefficients f_n^0 and f_n^c, some of which are reported in table 6. It would be interesting to be able to make the comparison.

6. Conclusions

Using environmentally friendly renormalization we derived a formal expression for the equation of state for the $O(N)$ model. This expression has the advantages that (i) it is *derived* from an underlying microscopic (field-theoretic) model; (ii) requires as input for any calculation only the three crossover Wilson functions associated with a magnetization-dependent renormalization of the field, φ, the composite operator, φ^2, and the coupling

constant, λ. In particular *no* experimental input is required in order to fix parameters; (iii) it is parameterized by two nonlinear scaling fields—the transverse mass, and an anisotropy parameter closely related to the stiffness constant; (iv) it manifestly expresses all relevant, desired analyticity properties, both in the critical region and on the coexistence curve; (v) universal coefficients associated with the expansion of the equation of state in *any* of the asymptotic regimes may be simply calculated thereby obtaining coefficients that are presently unknown.

After deriving one-loop expressions for general N, the formulation was then used to calculate a parameterized, analytic expression for the equation of state for $N = 1$ for $2 < d < 4$. For $d = 3$ it was shown how the parameterization could be dispensed with and a closed-form non-parameterized expression for the equation of state derived. Various universal coefficients associated with the asymptotic regimes $x = 0, x = \infty$ and $x = -1$ were derived for arbitrary d. For $d = 3$ these were compared with previous results.

By taking the functional form for the Wilson functions at one loop and introducing constants to fit to the best-known asymptotic values of the critical exponents, comparison was made for $d = 3$ and $d = 2$ between our calculated expansion coefficients and those, where known, as derived using HT expansions, Monte Carlo, etc, in the different asymptotic regimes. In most cases the fitted values were substantially better than the one-loop values, agreement being better for $d = 3$ than $d = 2$.

When continued to complex values of the external magnetic field the universal equation of state should also capture the Lee-Yang edge. There are additional points of non-analyticity in our expressions at $z_{\text{LY}}^2 = -2/(d - 2)$ or $z_{\text{LY}}^2 = -\nu/\beta$. These imaginary values of z are naturally associated with the Lee-Yang edge singularities. However, our equation of state is not yet optimized to include the associated crossover and we do not expect to obtain good estimates for the associated universal exponents or amplitudes. Our formulation can be adjusted to include this additional singularity, but as of yet we have not studied the effect of including this crossover. We hope to return to this in the future.

Acknowledgments

References

[1] Zinn-Justin J 2001 *Phys. Rep.* **344** 159
[2] Lawrie I D 1981 *J. Phys. A: Math. Gen.* **14** 24898
[3] Nienhuis B and Nauenberg M 1975 *Phys. Rev. Lett.* **35** 477
[4] Berker N and Fisher M E 1982 *Phys. Rev.* B **26** 2507
[5] Schäfer L and Horner H 1978 *Z. Phys.* B **29** 251
[6] Berges J Tetradis N and Wetterich C 1996 *Phys. Rev. Lett.* **77** 873
[7] Widom B 1965 *J. Chem. Phys.* **43** 3898
[8] Guida R and Zinn-Justin J 1997 *Nucl. Phys.* B **489** 626
[9] O'Connor D and Stephens C R 2002 *Phys. Rep.* **363** 425
     O'Connor D and Stephens C R 1994 *Int. J. Mod. Phys.* A **9** 2805
     O'Connor D and Stephens C R 1994 *Int. J. Mod. Phys.* A **9** 5851 (erratum)
     O'Connor D and Stephens C R 1991 *Nucl. Phys.* B **360** 297
[10] Kim J K and Landau D P 1997 *Nucl. Phys.* B **53** 706

[11] Pelissetto A and Vicari E 2002 *Phys. Rep.* **368** 549
     Campostrini M Pelissetto A Rossi P and Vicari E 2000 *Phys. Rev.* B **62** 5843
[12] Engels J Holtmann S Mendes T and Schulze T 2000 *Phys. Lett.* B **492** 219
[13] Schofield P 1969 *Phys. Rev. Lett.* **22** 606
[14] Brézin E Wallace D J and Wilson K G 1973 *Phys. Rev.* B **7** 232
[15] Wallace D J and Zia R K P 1975 *Phys. Rev.* B **12** 5340
[16] Griffiths R B 1967 *Phys. Rev.* **158** 176